

PHYSICS 523, GENERAL RELATIVITY
 Homework 1
 Due Wednesday, 27th September 2006
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Problem 1

a) We are to use the spacetime diagram of an observer \mathcal{O} to describe an ‘experiment’ specified by the problem 1.5 in Schutz’ text.

We have shown the spacetime diagram in Figure 1 below.

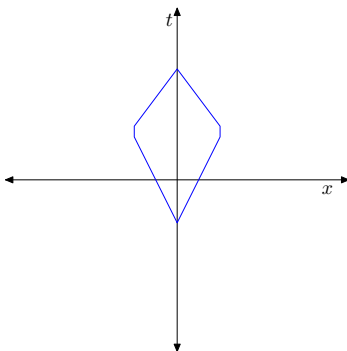


FIGURE 1. A spacetime diagram representing the experiment which was required to be described in Problem 1.a.

b) The experimenter observes that the two particles arrive back at the same point in spacetime after leaving from equidistant sources. The experimenter argues that this implies that they were released ‘simultaneously;’ comment.

In his frame, his reasoning is just, and implies that *his* t -coordinates of the two events have the same value. However, there is no absolute simultaneity in spacetime, so a different observer would be free to say that in *her frame*, the two events were not simultaneous.

c) A second observer $\overline{\mathcal{O}}$ moves with speed $v = 3c/4$ in the *negative* x -direction relative to \mathcal{O} . We are asked to draw the corresponding spacetime diagram of the experiment in this frame and comment on simultaneity.

Calculating the transformation by hand (so the image is accurate), the experiment observed in frame $\overline{\mathcal{O}}$ is shown in Figure 2. Notice that observer $\overline{\mathcal{O}}$ does not see the two emission events as occurring simultaneously.

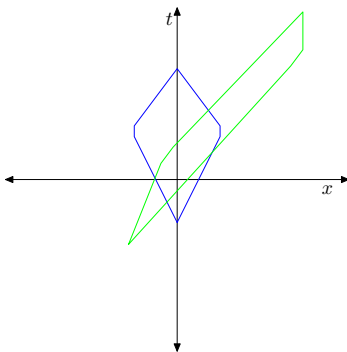


FIGURE 2. A spacetime diagram representing the experiment in two different frames. The worldlines in blue represent those recorded by observer \mathcal{O} and those in green represent the event as recorded by an observer in frame $\overline{\mathcal{O}}$. Notice that there is obvious ‘length contraction’ in the negative x -direction and time dilation as well.

d) We are to show that the invariant interval between the two emission events is invariant.

We can proceed directly. It is necessary to know that in frame \mathcal{O} the events have coordinates $p_1 = (5/2, -2)$ and $p_2 = (5/2, 2)$ while in frame \mathcal{O}' they have coordinates $\bar{p}_1 = \gamma(1, -1/8)$ and $\bar{p}_2 = \gamma(4, 31/8)$ where $\gamma^2 = \frac{16}{7}$.

$$\Delta s^2 = (p_1 - p_2)^2 = 16;$$

$$\Delta \bar{s}^2 = (\bar{p}_1 - \bar{p}_2)^2 = \gamma^2(-9 + 16) = 16.$$

We see that the invariant interval is indeed invariant in this pointless example.

Problem 2.

a) We are to show that rapidity is additive upon successive boosts in the same direction.

We may as well introduce the notation used in the problem: let $v = \tanh \alpha$ and $w = \tanh \beta$; this allows us to write $\gamma = \frac{1}{\sqrt{1-\tanh^2 \alpha}} = \cosh \alpha$ and $v\gamma = \sinh \alpha$, and similar expressions apply for β . We see that using this language, the boost transformations are realized by the matrices¹

$$\begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \mapsto \begin{pmatrix} \cosh \alpha & -\sinh \alpha \\ -\sinh \alpha & \cosh \alpha \end{pmatrix}, \quad (\text{a.1})$$

and similarly for the boost with velocity w . Two successive boosts are then composed by²:

$$\begin{aligned} \begin{pmatrix} \cosh \beta & -\sinh \beta \\ -\sinh \beta & \cosh \beta \end{pmatrix} \begin{pmatrix} \cosh \alpha & -\sinh \alpha \\ -\sinh \alpha & \cosh \alpha \end{pmatrix} &= \begin{pmatrix} \cosh \alpha \cosh \beta + \sinh \alpha \sinh \beta & -\sinh \alpha \cosh \beta - \cosh \alpha \sinh \beta \\ -\sinh \alpha \cosh \beta - \cosh \alpha \sinh \beta & \cosh \alpha \cosh \beta + \sinh \alpha \sinh \beta \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\alpha + \beta) & -\sinh(\alpha + \beta) \\ -\sinh(\alpha + \beta) & \cosh(\alpha + \beta) \end{pmatrix} \end{aligned}$$

This matrix is itself a boost matrix, now parameterized by a rapidity parameter $(\alpha + \beta)$. Therefore, successive boosts are additive for rapidity.

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b) Consider a star which observes a second star receding at speed $9c/10$; this star measures a third moving in the same direction, receding with the same relative speed; this build up continues consecutively N times. What is the velocity of the N^{th} star relative to the first? Give the explicit result for all N .

From the additivity of the rapidity, we see immediately that $\eta' = N\eta$ where $\eta = \operatorname{arctanh}(9/10)$ and η' is the rapidity of the resulting velocity. That is $\eta' = \operatorname{arctanh}(\beta)$ where β is the recession velocity of the N^{th} star relative to the first.

Recall a nice identity easily obtainable from the canonical definitions of $\tanh(x)$:

$$\operatorname{arctanh}(x) = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right). \quad (\text{a.2})$$

Therefore, we have that

$$\begin{aligned} \log \left(\frac{1+\beta}{1-\beta} \right) &= N \log \left(\frac{1+\frac{9}{10}}{1-\frac{9}{10}} \right), \\ &= \log(19^N), \\ \therefore \frac{1+\beta}{1-\beta} &= 19^N, \\ \boxed{\therefore \beta = \frac{19^N - 1}{1 + 19^N}} & \quad (\text{a.3}) \end{aligned}$$

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¹Because we are considering successive boosts in one direction, the problem really lives in $1+1$ -dimensions and we can make life easier with only 2×2 matrices.

²We make use of some obvious identities in hyperbolic geometry.

Problem 3.

a) Consider a boost in the x -direction with speed $v_A = \tanh \alpha$ followed by a boost in the y -direction with speed $v_B = \tanh \beta$. We are to show that the resulting Lorentz transformation can be written as a pure rotation followed by a pure boost and determine the rotation and boost.

This is a 2 + 1-dimensional problem—the entire problem involves only the $SO(2, 1)$ subgroup of the Lorentz group. Now, although there must certainly be easy ways of solving this problem without setting up a system of equations and using trigonometric identities, we will stick with the obvious answer/easy math route—indeed, the algebra is not that daunting and the equations are easily solved.

The brute-force technique involves writing out the general matrices for both operations and (consistently) matching terms. The two successive boosts result in

$$\begin{pmatrix} \cosh \beta & 0 & -\sinh \beta \\ 0 & 1 & 0 \\ -\sinh \beta & 0 & \cosh \beta \end{pmatrix} \begin{pmatrix} \cosh \alpha & -\sinh \alpha & 0 \\ -\sinh \alpha & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh \alpha \cosh \beta & -\sinh \alpha \cosh \beta & -\sinh \beta \\ -\sinh \alpha & \cosh \alpha & 0 \\ -\cosh \alpha \sinh \beta & \sinh \alpha \sinh \beta & \cosh \beta \end{pmatrix} \quad (\text{a.4})$$

And a rotation about \hat{z} through the angle θ followed by a boost in the $(\cos \lambda, \sin \lambda)$ -direction with rapidity η is given by³

$$\begin{pmatrix} \cosh \eta & -\sinh \eta \cos \lambda & -\sinh \eta \sin \lambda \\ -\sinh \eta \cos \lambda & 1 + \cos^2 \lambda (\cosh \eta - 1) & \cos \lambda \sin \lambda (\cosh \eta - 1) \\ -\sinh \eta \sin \lambda & \cos \lambda \sin \lambda (\cosh \eta - 1) & 1 + \sin^2 \lambda (\cosh \eta - 1) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \\ = \begin{pmatrix} \cosh \eta & -\sinh \eta (\cos \lambda \cos \theta + \sin \lambda \sin \theta) & -\sinh \eta (\sin \lambda \cos \theta - \cos \lambda \sin \theta) \\ -\sinh \eta \cos \lambda & \cos \theta + (\cosh \eta - 1)(\cos^2 \lambda \cos \theta + \cos \lambda \sin \lambda \sin \theta) & -\sin \theta + (\cosh \eta - 1)(\cos \lambda \sin \lambda \cos \theta - \cos^2 \lambda \sin \theta) \\ -\sinh \eta \sin \lambda & \sin \theta + (\cosh \eta - 1)(\sin^2 \lambda \sin \theta + \cos \lambda \sin \lambda \cos \theta) & \cos \theta + (\cosh \eta - 1)(\sin^2 \lambda \cos \theta - \cos \lambda \sin \lambda \sin \theta) \end{pmatrix}$$

The system is over-constrained, and it is not hard to find the solutions. For example, the (00)-entry in both transformation matrices must match,

$$\boxed{\therefore \cosh \eta = \cosh \alpha \cosh \alpha.} \quad (\text{a.5})$$

Looking at the (10) and (20) entries in each box, we see that

$$\begin{aligned} \sinh \eta \cos \lambda &= \sinh \alpha; \\ \sin \eta \sin \lambda &= \cosh \alpha \sinh \beta; \end{aligned}$$

which together imply

$$\boxed{\therefore \tan \lambda = \frac{\sinh \beta}{\tanh \alpha}.} \quad (\text{a.6})$$

Lastly, we must find θ ; this can be achieved via the equation matching for the (12) entry:

$$\begin{aligned} \sin \theta &= (\cosh \eta - 1) (\cos \lambda \sin \lambda \cos \theta - \cos^2 \lambda \sin \theta); \\ \implies \tan \theta (1 + \cos^2 \lambda (\cosh \eta - 1)) &= (\cosh \eta - 1) \cos \lambda \sin \lambda, \end{aligned}$$

$$\boxed{\therefore \tan \theta = \frac{(\cosh \eta - 1) \cos \lambda \sin \lambda}{1 + \cos^2 \lambda \cosh \eta - \cos^2 \lambda}.} \quad (\text{a.7})$$

b) A spaceship A moves with velocity v_A along \hat{x} relative to \mathcal{O} and another, B , moves with speed v_B along \hat{y} relative to A . Determine the direction and velocity of the frame \mathcal{O} relative to B .

To map this exactly to the previous problem, we do things backwards and transform $B \rightarrow A$ followed by $A \rightarrow \mathcal{O}$ relative to B . That is, let $\tanh \alpha = -v_B$ and $\tanh \beta = v_A$. Now, the magnitude of the velocity of frame \mathcal{O} relative to B has rapidity given by equation (a.5), and is moving in the direction an angle $\pi - (\theta + \lambda)$ relative to A where λ and θ are given by equations (a.6) and (a.7), respectively.

³This required a bit of algebra, but it isn't worth doing in public.